Applications of Ampere's Law

In electrostatics, the electric field due to any known charge distribution $\rho(x, y, z)$ may always be obtained from the Coulomb law — it's a universal tool — but the actual calculation is often rather difficult. By comparison, the Gauss law can gives us the electric fields of a sufficiently symmetric charge configuration on a silver platter, without any hard work, — but it would not do us much good without a symmetry. In magnetostatics, the role of the Coulomb law is played by the Biot–Savart–Laplace law: it works for all steady currents, but the actual calculation can be quite painful. By the same analogy, the Ampere's law

$$\oint_{\mathcal{L}} \mathbf{B} \cdot d\vec{\ell} \;=\; \mu_0 \times I_{\text{net}}(\text{through loop } \mathcal{L}) \tag{1}$$

plays the role of the Gauss law: It can give us the magnetic field $\mathbf{B}(x, y, z)$ of a sufficiently symmetric system of currents without taking any integrals, but the symmetry is essential for any useful Ampere's-law based calculation.

These notes give several examples of using the Ampere's law to find the magnetic field.

Long Straight Wire

In the first example we reproduce the well-known magnetic field of an infinite straight wire from the Ampere's law. We begin by using the symmetries of the wire — which we take to run along the z axis — to find the direction of the magnetic field. First, consider mirror reflection from the xz plane, $(x, y, z) \to (+x, -y, +z)$. Since the magnetic field is an axial vector rather than a true vector, its components transform under this reflection according to

$$B_x \;\to\; -B_x\,, \quad B_y \;\to\; +B_y\,, \quad B_z \;\to\; -B_z\,, \tag{2}$$

At the points invariant under this reflection, the magnetic field must also be invariant, thus in the xz plane where $y = 0$, the magnetic field points in the $= \pm\hat{\mathbf{y}}$ direction — which happens to be the direction of a circle surrounding the wire. Consequently, by the rotational symmetry around the z axis, *the magnetic field everywhere points in the direction of the circle around the wire;* in the cylindrical coordinates (s, ϕ, z) this is the $\hat{\phi}$ direction.

Also, the rotational symmetry makes the magnitude B ϕ-independent, while the translation symmetry in z direction makes it z-independent. Altogether,

$$\mathbf{B}(s, \phi, z) = B(s \text{ only}) \, \hat{\phi}. \tag{3}$$

Given this symmetry-restricted from of the magnetic field, we may find its radius dependence from the Ampere's law. Let the Ampere's loop $\mathcal{L}$ be a circle in the plane $\perp$ to the wire and centered on the wire,

(4)
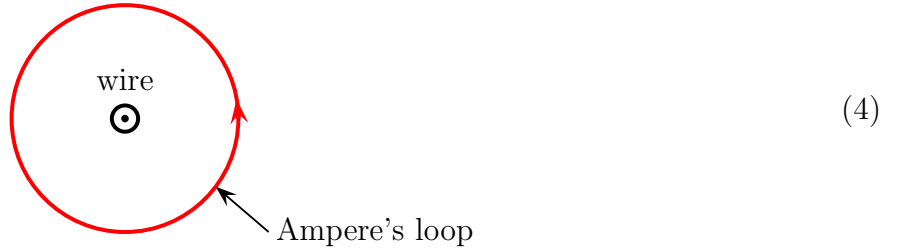
For this loop, $d\vec{\ell} = s \, d\phi \, \hat{\phi}$, hence

$$\oint \mathbf{B} \cdot d\vec{\ell} = \oint B(s) \times s \times d\phi = 2\pi s \times B(s). \tag{5}$$

On the other hand, the current through the loop is simply the net current I in the wire, thus

$$2\pi s \times B(s) = \mu_0 \times I \quad \Longrightarrow \quad B(s) = \frac{\mu_0 \times I}{2\pi s}. \tag{6}$$

Now consider a wire of finite thickness. For simplicity, let the wire have round cross-section of radius a and uniform current density

$$\mathbf{J} = \frac{I}{\pi a^2} \, \hat{\mathbf{z}}. \tag{7}$$

In this case, the same considerations as for the thin wire tells us that the magnetic field is in the $\hat{\phi}$ direction while its magnitude depends only on the radial coordinate s. Moreover,

the magnitude follows from the same kind of Ampere's loop, a circle centered on the wire's axis (4), so the Ampere's law tells us that

$$2\pi s \times B(s) = \mu_0 \times I(\text{through loop of radius } s). \tag{8}$$

For $s > a$, the loop is outside the wire and circles around it, so the current through the loop is the net current I, hence

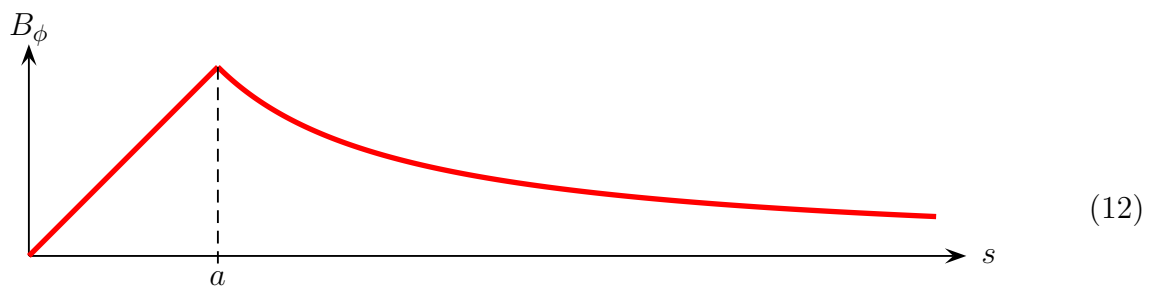$$\mathbf{B}^{\text{outside}} = \frac{\mu_0 I}{2\pi s} \hat{\phi}, \tag{9}$$

exactly as for the thin wire! On the other hand, for $s < a$, the loop is inside the wire, so the current through the loop is in propoertion to its area,

$$I(\text{through the loop}) = J \times \pi s^2 = I \times \frac{s^2}{a^2}, \tag{10}$$

and therefore

$$\mathbf{B}^{\text{inside}} = \frac{\mu_0 I}{2\pi} \frac{s}{a^2} \hat{\phi}. \tag{11}$$

Here is the profile of the magnetic field strength as a function of the radial coordinate s,

$$\tag{12}$$

For our next example, consider infinite 2D current sheet spanning the xy plane, with a uniform 2D current density

$$K = \frac{I}{\Delta x} \tag{13}$$

in the $\hat{\mathbf{y}}$ direction. This current is invariant under translations in x and y directions, so the magnetic field $\mathbf{B}(z)$ depends only on the z coordinate but not on x or y. To establish the field's direction, we use another symmetry, the reflection from the yz plane,

$$(x, y, z) \rightarrow (-x, +y, +z), \quad B_x \rightarrow +B_x, \quad B_y \rightarrow -B_y, \quad B_z \rightarrow -B_z, \tag{14}$$

which tells us that the magnetic field points along the x axis, parallel to the current sheet but perpendicular to the current. Thus,

$$\mathbf{B}(x, y, z) = B(z \text{ only}) \, \hat{\mathbf{x}}. \tag{15}$$

In a moment, I shall obtain the $B(z)$ function here from the Ampere's law, but first let me use yet another symmetry — the 180° rotation about the y axis — to establish that the $B(z)$ function is odd,

$$B(-z) = -B(+z). \tag{16}$$

In other words, the magnetic field above the current sheet points in the opposite direction from the field below the sheet.

At this point, let's use the Ampere law for a rectangular $\Delta x \times \Delta z$ loop,

(17)
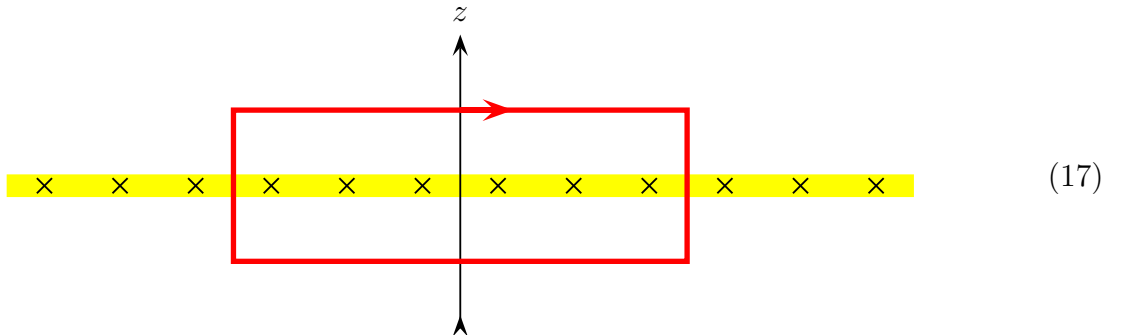
For this loop,

$$\oint \mathbf{B} \cdot d\vec{\ell} \;=\; \Delta x \times B_x(\text{top}) \;-\; \Delta x \times B_x(\text{bottom}), \tag{18}$$

and if we put the top of the loop at $+z$ while the bottom of the loop is at $-z$, then by the antisymmetry (16),

$$\oint \mathbf{B} \cdot d\vec{\ell} \;=\; 2 \times \Delta x \times B(z). \tag{19}$$

At the same time, the current through the loop is the current through width Δx of the current sheet, thus

$$I(\text{through the loop}) \;=\; K \times \Delta x, \tag{20}$$

hence by the Ampere's law

$$2 \times \Delta x \times B(z) \;=\; \mu_0 \times K \times \Delta x \tag{21}$$

and therefore

$$B(z) \;=\; \frac{\mu_0 K}{2} \quad \text{for any } z > 0. \tag{22}$$

Altogether, the magnetic field of the current sheet is

$$\mathbf{B}(x, y, z) \;=\; \begin{cases} +\dfrac{\mu_0 K}{2}\, \hat{\mathbf{x}} & \text{above the current sheet } (z > 0), \\[2ex] -\dfrac{\mu_0 K}{2}\, \hat{\mathbf{x}} & \text{below the current sheet } (z < 0). \end{cases} \tag{23}$$

Note that the field is completely uniform above the sheet, and likewise uniform below the sheet, but across the sheet there is discontinuity

$$\text{disc}(\mathbf{B}) \;=\; \mathbf{B}^{\text{above}} \;-\; \mathbf{B}^{\text{below}} \;=\; \mu_0 K\, \hat{\mathbf{x}} \;=\; \mu_0 \mathbf{K} \times \mathbf{n} \tag{24}$$

where $\mathbf{n}$ is the unit vector $\perp$ to the current sheet, in out case $\mathbf{n} = \hat{\mathbf{z}}$.

LONG SOLENOID

For the next example, consider a *solenoid* where the wire is coiled around a cylinder, like on the picture below:

Each wire loop in this coil acts as if it was a separate circular wire carrying the same current I — the current in the coiled wire — and the dense enough array of such loops may be approximated by a *cylindrical current sheet* with the current density

$$K \;=\; I \times \frac{N}{L} \;=\; I \times \frac{N_{\text{loops}}}{\text{solenoid length}} \tag{25}$$

in the $\hat{\phi}$ direction around the solenoid.

For simplicity, let's assume a long solenoid (length $\gg$ diameter) which we approximate as infinitely long. This gives us translational symmetry in the z direction and hence z-independent magnetic field. Also, the rotational symmetry around the z axis makes the field ϕ-independent, thus $\mathbf{B} = \mathbf{B}(s \text{ only})$. To find the direction of this field, we use yet another symmetry, the reflection from a mirror in the xy plane,
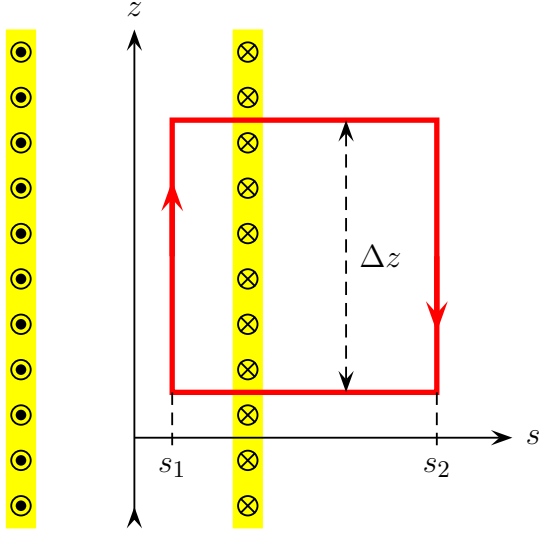
$$(x, y, z) \;\to\; (+x, +y, -z), \quad B_x \;\to\; -B_x, \quad B_y \;\to\; -B_y, \quad B_z \;\to\; +B_z, \tag{26}$$

which requires the magnetic field *everywhere* to point along the z axis. Altogether,

$$\mathbf{B}(s, \phi, z) \;=\; B(s)\,\hat{\mathbf{z}}. \tag{27}$$

To find the radial dependence of this field, we may use the Ampere's law. Consider a

rectangular loop in the zx plane such that one side of the loop is inside the solenoid while the other side is outside, such as on this cross-sectional picture:

$$\oint \mathbf{B} \cdot d\vec{\ell} = \Delta z \times B(s_1) - \Delta z \times B(s_2). \qquad (28)$$

The net current through this loop includes all the solenoid's coils crossing the loop (28), thus

$$I(\text{through the loop}) = I \times N \times \frac{\Delta z}{L}, \qquad (29)$$

or in terms of the surface current density (25),

$$I(\text{through the loop}) = K \times \Delta z. \qquad (30)$$

Hence, by the Ampere's law,

$$\Delta z \times \big(B(s_1) - B(s_2)\big) = \mu_0 \times \frac{IN}{L} \times \Delta z \qquad (31)$$

and therefore

$$B(s_1) - B(s_2) = \frac{\mu_0 IN}{L}. \qquad (32)$$

Note that the RHS of this formula does not depend on the radii s_1 and s_2 as long as s_1 is inside the solenoid while s_2 is outside the solenoid. Consequently, the magnetic field inside the solenoid is uniform; the magnetic field outside the solenoid is also uniform, but there is a discontinuity across the solenoid's wiring.

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Finally, very far from the solenoid, the magnetic field must diminish to zero. At the same time, the field outside the solenoid must be uniform, which means *zero magnetic field everywhere outside the solenoid!* Consequently, in light of eq. (32), *the magnetic field inside the solenoid is uniform*

$$\mathbf{B} \;=\; \frac{\mu_0 I N}{L}\,\hat{\mathbf{z}}\,. \tag{33}$$

What about solenoids of finite length? In this case, the magnetic field has both B_z and B_s components and depends on both s and z coordinates. Alas, the exact formulae for the $B_z(s,z)$ and $B_s(s,z)$ are way too complicated for this class. Instead, let me give you the qualitative picture of the magnetic field lines:
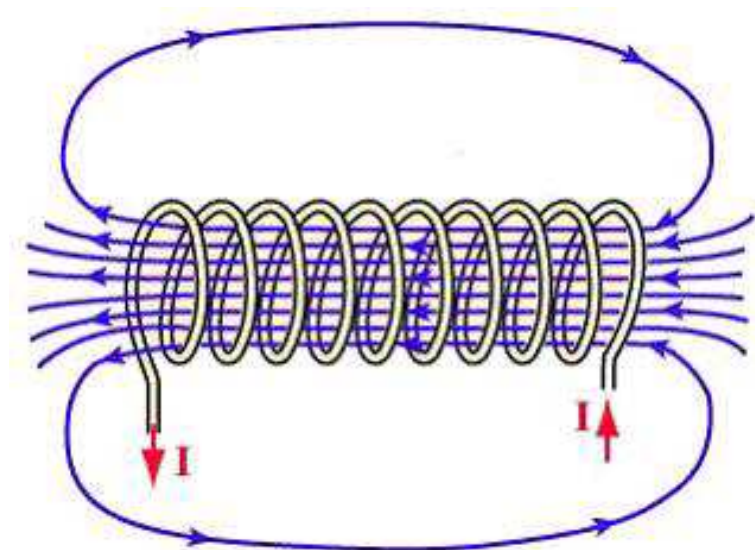

For a long solenoid (compared to its diameter), the magnetic field inside the solenoid is *approximately* uniform and *approximately* parallel to the axis, except near the ends of the solenoid. Outside the solenoid, the magnetic field looks like the field of a physical dipole, with the North pole at one end of the solenoid and the South pole at the other end.

TOROIDAL COIL

For our final example, consider a wire coil wound around a *torus* rather than a cylinder:

More generally, instead of a bagel-shaped torus we may use a toroid with a square cross-section like
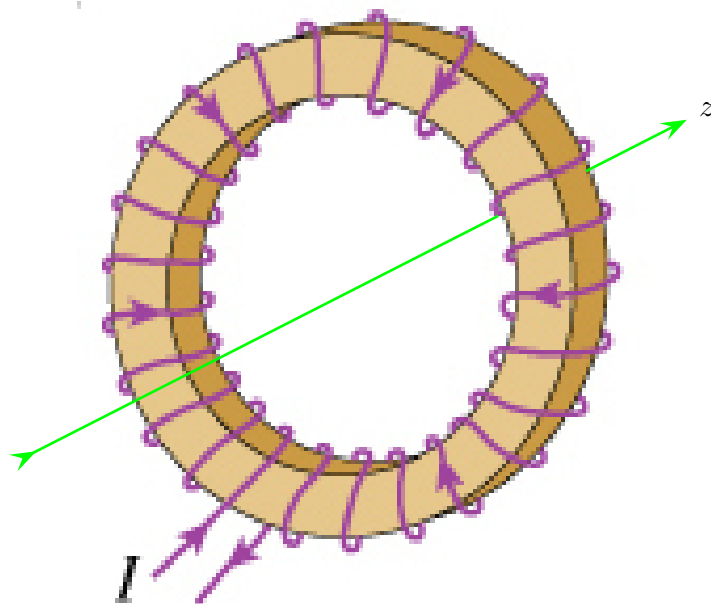
or any other cross-sectional shape, as long there is a rotational symmetry around the central axis z. The winding should be uniform in the $\hat{\phi}$ direction, which means that its density per

unit length,

$$\frac{N}{L} = \frac{N}{2\pi R} \tag{34}$$

is larger on the inner side of the toroid than on the outer side, simply because of the shorter length $L = 2\pi R$.

Let's find the magnetic field of this toroidal coil. Combining the rotational symmetry around the z axis with a mirror reflection about some cross-sectional plane such as xz, we find that the magnetic field everywhere must point in the $\hat{\phi}$ direction; thus, the magnetic field lines form circles around the central axis z. Also, the magnitude of the magnetic field does not depend on ϕ, also it may depend on the other two cylindrical coordinates s and z. Altogether, the symmetries restrict the magnetic field to

$$\mathbf{B}(s, \phi, z) = B(s, z \text{ only}) \, \hat{\phi}. \tag{35}$$

To find the dependence of this field on the s and z coordinates, we use the Ampere's law. This time, the Ampere loop is a circle centered on z axis; in cylindrical coordinates, it has fixed s and z while spanning ϕ from 0 to 2π. This loop follows a magnetic field line, and since the field is uniform along this line, the Ampere integral is simply

$$\oint_{\text{loop}} \mathbf{B} \cdot d\vec{\ell} = 2\pi s \times B(s, z). \tag{36}$$

As to the net current through this loop, *If the loop lies inside the toroidal coil,*
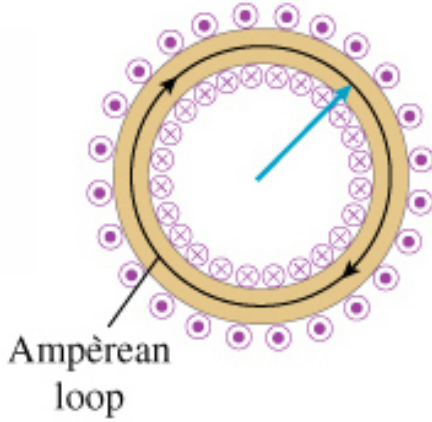
$$I_{\text{net}}(\text{through Ampere loop}) = N \times I \tag{37}$$

since every winding of the coil goes through the loop and their currents add up. On the other hand, if the Ampere loop lies outside the coil, there is no current at all through that loop. Consequently, by the Ampere's law,

$$\begin{aligned} \text{Inside the coil,} \quad \mathbf{B} &= \frac{NI\mu_0}{2\pi s} \, \hat{\phi}, \\ \text{Outside the coil,} \quad \mathbf{B} &= 0. \end{aligned} \tag{38}$$